W25. For $n \in \mathbb{N}$, with notation,

$$r_n = \sqrt{n \cdot \sqrt{(n-1) \cdot \sqrt{(n-2) \cdot \ldots \cdot \sqrt{2 \cdot \sqrt{1}}}}}\ ,$$

prove that :

a) $r_n \le \left(\dfrac{(n-1)2^n + 1}{2^n - 1}\right)^{1-\frac{1}{2^n}}$ b) $\dfrac{r_n}{n} \in (0, 1]$.

Dorin Mărghidanu.

Solution by Arkady Alt, **San Jose**, **California**, **USA.**

By AM-GM Inequality we have

a) $$r_n = n^{1/2} \cdot (n-1)^{1/2^2} \cdot (n-2)^{1/2^3} \cdot \ldots \cdot 2^{1/2^{n-1}} \cdot 1^{1/2^n} \le \left(\dfrac{\sum_{k=1}^{n} \dfrac{n-k+1}{2^k}}{\sum_{k=1}^{n-1} \dfrac{1}{2^k}}\right)^{\sum_{k=1}^{n-1} \frac{1}{2^k}}.$$

Since $\sum_{k=1}^{n} \dfrac{1}{2^k} = \dfrac{2^n - 1}{2^n}$ and* $\sum_{k=1}^{n} \dfrac{n-k+1}{2^k} = n\sum_{k=1}^{n} \dfrac{1}{2^k} - \sum_{k=1}^{n} \dfrac{k-1}{2^k} =$

$\dfrac{(n+1)(2^n-1)}{2^n} - \left(1 - \dfrac{n+1}{2^n}\right) = \dfrac{2^n(n-1)+1}{2^n}$ then $r_n \le \left(\dfrac{(n-1)2^n + 1}{2^n - 1}\right)^{1-\frac{1}{2^n}}$.

b) $$\frac{r_n}{n} = \sqrt{1 \cdot \sqrt{\left(1-\frac{1}{n}\right) \cdot \sqrt{\left(1-\frac{2}{n}\right) \cdot \ldots \cdot \sqrt{\left(1-\frac{n-2}{n}\right) \cdot \sqrt{\left(1-\frac{n-1}{n}\right)}}}}} < 1.$$

* Since $\sum_{k=1}^{n} kx^{k-1} = \dfrac{1-(n+1)x^n + nx^{n+1}}{(1-x)^2}$ then $\sum_{k=1}^{n} \dfrac{k-1}{2^k} = \sum_{k=2}^{n} \dfrac{k-1}{2^k} =$

$\sum_{k=1}^{n-1} \dfrac{k}{2^{k+1}} = \dfrac{1}{4}\sum_{k=1}^{n-1} \dfrac{k}{2^{k-1}} = 1 - \dfrac{n+1}{2^n}$.